symmetric, and those of the last row to the value of n for which Ref. 5 leads to a minimum (namely $n = 3^{1/2}/2$, 0.50, 0.90 and $3^{1/2}/2$ for the four sections, respectively). As expected, Eq. (6) gives too conservative values, and so does Eq. (10b) in the case of section B. For sections A, C, and D, the ratios of the values for the minimum ΔT_{cr} given in the last two rows are 0.70, 0.92, and 0.69, respectively; these may be considered sufficiently close to unity to be useful either for quick preliminary estimates or for rough checks on more accurate calculations. It may be indeed concluded that this will always be the case for all sections, except those in which the bulk of the material is concentrated, as in section B, away from the tip regions $|r| > \rho$.

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A Finite Element Solution for Saint-Venant Bending

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Nomenclature¶

A,a = area of cross section, area of element
$E_{(x,y)}$ = modulus of elasticity in the z direction
G_{xz},G_{yz} = torsional moduli on x and y axes of orthotropy
K_{1e},K_{2e} = curvature defined in Eq. (6)
P_x,P_y = end loads in the x and y direction applied at the shear center (Fig. 1)
$w_z,\bar{w},w$ = warping functions [Eq. (9) and (14)]
σ,ϵ = normal stress and strain
τ,γ = shearing stress and strain
ν_{zx},ν_{zy} = Poisson's ratio for strain in the x direction due to stress in the z direction, etc.

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¶ For definitions of properties of cross section such as A_e, M_{xe}, etc., see Eq. (5). See also basic nomenclature of Ref. 9.

1. Introduction

FOR moderately thick beams, the classical Euler-Bernoulli beam theory must be modified to include shear deformation, especially when such a beam is vibrating at frequencies above the fundamental. The equations which include this shear effect as well as rotatory inertia are due to Timoshenko.[1] In these equations, the shearing strain may be expressed as the average shearing stress divided by the shear modulus and a shear coefficient K. This dimensionless coefficient, which depends on the shape of the crosssection, must be introduced since neither the shearing stress nor the shearing strain are uniformly distributed over the crosssection.

A recent contribution[2] on evaluating an acceptable value of K requires an elasticity displacement solution for bending of beams subjected to end load. Saint-Venant solved this problem for a homogeneous isotropic beam. Recent solutions for isotropic and aeolotropic material, in terms of a stress function have been given by several authors.[3–5] The problem has also been solved for beams of nonhomogeneous material whose Poisson's ratio is constant.[6,7] Beams of only a few simple geometric sections have been solved exactly due to the mathematical complexity involved. Thus, an approximate solution is needed. A numerical solution will yield the transverse shear-stress distribution, the associated longitudinal displacements (warping), and the location of the shear center.

2. Saint-Venant Beam Theory for Nonhomogeneous Orthotropic Beams

The method of solution is to assume that the stresses σ_x, σ_y, and τ_{xy}, in the beam of Fig. 1 are zero and then show that, for either the isotropic or the orthotropic homogeneous beam, four of the compatibility equations can be satisfied if ϵ_z is of the form,

$$\epsilon_z = c_1zy + c_2zx + c_3x + c_4y + c_5z + c_6 \quad \sigma_z = E_{(x,y)}\epsilon_z \quad (1)$$

where c_1–c_6 are constants of integration. Equation (1) is also true for isotropic or orthotropic nonhomogeneous beams when all Poisson's ratios are constant. Boundary conditions to be satisfied are

$$\int_A \sigma_z dA = 0 \quad \int_A \sigma_z x dA = -P_x(l - z)$$
$$\int_A \sigma_z dA = -P_y(l - z) \quad (2)$$

If Eqs. (2) are satisfied then the boundary conditions

$$\int_A \tau_{xz} dA = P_x \quad \int_A \tau_{yz} dA = P_y \quad (3)$$

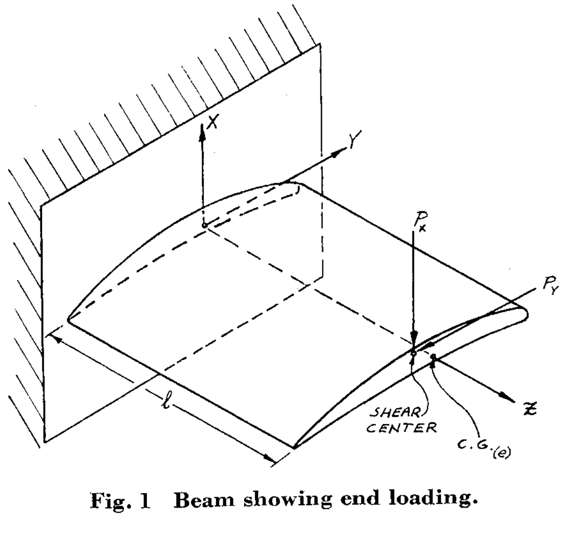

Fig. 1 Beam showing end loading.

Table 1 Comparison of numerical results to the exact solution for shear coefficient of isotropic homogeneous square and circular sections

Square No. elements	K	$-\nu = \frac{1}{3}-$ Circle No. elements	K
4	1.015	76	0.9007
121	0.8586	152	0.8989
256	0.8547	Exact	0.8889
Exact	0.8510		

will also be satisfied. Substituting Eq. (1) into Eq. (2) yields

$$c_1N = P_x(M_{xe}M_{ye} - A_eI_{xye}) + P_y(I_{ye}A_e - M_{ye}^2) \quad (4a)$$

$$c_2N = P_x(I_{xe}A_e - M_{xe}^2) + P_y(M_{xe}M_{ye} - I_{xye}A_e) \quad (4b)$$

$$c_3N = lP_x(M_{xe}^2 - I_{xe}A_e) + lP_y(I_{xye}A_e - M_{xe}M_{ye}) \quad (4c)$$

$$c_4N = lP_x(I_{xye}A_e - M_{xe}M_{ye}) + lP_y(-I_{ye}A_e + M_{ye}^2) \quad (4d)$$

$$c_5N = P_x(I_{xye}M_{xe} - I_{xe}M_{ye}) + P_y(I_{xye}M_{ye} - I_{ye}M_{xe}) \quad (4e)$$

$$c_6N = lP_x(I_{xe}M_{ye} - I_{xye}M_{xe}) + lP_y(I_{ye}M_{xe} - I_{xye}M_{ye}) \quad (4f)$$

where $N = (I_{xe}I_{ye} - I_{xye}^2)A_e - I_{xe}M_{ye}^2 - I_{ye}M_{xe}^2 + 2I_{xye}M_{xe}M_{ye}$,

$$A_e = \textstyle\int_A EdA \quad M_{xe} = \int_A yEdA \quad M_{ye} = \int_A xEdA$$

$$I_{xe} = \textstyle\int_H y^2EdA \quad I_{ye} = \int_A x^2EdA \quad I_{xye} = \int_A xyEdA \quad (5)$$

If the x and y axes are located such that $M_{xe} = M_{ye} = 0$ (i.e., area-material center of gravity is zero) then

$$c_2 = -c_3/l = K_{1e} \quad c_1 = -c_4/l = K_{2e} \quad c_5 = c_6 = 0 \quad (6)$$

where

$$K_{2e} = \frac{-P_xI_{xye} + P_yI_{ye}}{I_{xe}I_{ye} - I_{xye}^2} \quad K_{1e} = \frac{P_xI_{xe} - P_yI_{xye}}{I_{xe}I_{ye} - I_{xye}^2} \quad (7)$$

and

$$\sigma_z = -E(l - z)(K_{1e}x + K_{2e}y) \quad (8)$$

Equation (8) shows that in the Saint-Venant bending problem σ_z is distributed in the same manner as in the case of pure bending. For homogeneous isotropic or orthotropic material and x and y centroidal axes, Eqs. (7) and (8) reduce to those given in Art. 52 of Ref. 5.

Two of the equations of equilibrium and the strain displacement equations yield the z displacement and the shearing strains as

$$w_z = -(lz - z^2/2)(K_{1e}x + K_{2e}y) + \bar{w}_{(x,y)} \quad (9)$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} - \nu_{zx}[K_{2e}xy + K_{1e}x^2/2] + \nu_{zy}K_{1e}y^2/2$$

$$\tau_{xz} = G_{xz}\gamma_{xz} \quad (10a)$$

$$\gamma_{yz} = \frac{\partial \bar{w}}{\partial y} - \nu_{zy}[K_{2e}y^2/2 + K_{1e}xy] + \nu_{zx}K_{2e}x^2/2$$

$$\tau_{yz} = G_{yz}\gamma_{yz} \quad (10b)$$

For the nonhomogeneous beam, Eqs. (8), (23), and (29) of Ref. 2 become

$$\Phi = \frac{1}{I_{ye}}\iint_A xEu_zdxdy \quad \frac{\partial\Phi}{\partial z} = \frac{M}{I_{ye}}, \quad \frac{\partial W}{\partial z} + \Phi = \frac{P_x}{KA} \quad (11)$$

where

$$K = I_{ye}/[(\nu_{zy}I_x - \nu_{zx}I_y)/2 + A\textstyle\iint_A xE\bar{w}dA/P_x$$

The shear coefficient K in Eqs. (11) is no longer dimensionless but has the units of stress. Note that Eqs. (9–11) contain $\bar{w}$ as the only unknown quantity.

3. Finite Element Formulation

The finite element method appears to be the most suitable to solve for $\bar{w}_{(x,y)}$ and hence the entire bending problem.[8] The method of Ref. 8 obtains a displacement formulation by a finite element technique which combines the total potential energy of all elements adjacent to a given node. The energy is then minimized to obtain a relation between the displacement at a given node and that of the surrounding nodes. The solution of Ref. 8 applies only for homogeneous isotropic material.

The present Note presents a finite element representation of the general Saint-Venant bending problem by a direct stiffness method employed in solving the torsion problem.[9] The element of Fig. 1 of Ref. 9 is utilized, however, in addition to the existing loads on this element, the two surfaces perpendicular to the z axis, in the corresponding bending element, are stressed an amount of σ_z and $(\sigma_z + \Delta\sigma_z)$. In the direct stiffness method** the element distributed surface forces τ_{xz}, τ_{yz}, and σ_z are replaced by discrete corner forces Z_i, such that the virtual work due to all the surface distributed forces is equal to the virtual work due to the Z_i system of forces. This equality expressed as

$$\sum_{i=1}^{k} Z_i\delta\bar{w}_i = \iint_a [\tau_{xz}(\delta\tau_{xz}) + \tau_{yz}(\delta\tau_{yz}) - E(K_{1e}x + K_{2e}y)]\delta\bar{w}]da \quad (12)$$

is employed to derive the element stiffness matrix.

Consider a homogeneous and orthotropic triangular element whose vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) are numbered counterclockwise. The stiffness matrix for such a triangular element will now be derived.

Let the warping function for the triangular element be approximated by the linear relation

$$w = w_1[(x - x_2)Y_{23} - (y - y_2)X_{23}]/2a + w_2[(x - x_3)Y_{31} - (y - y_3)X_{31}]/2a + w_3[(x - x_1)Y_{12} - (y - y_1)X_{12}]/2a \quad (13)$$

where $X_{ij} = x_i - x_j$ and w_1,w_2,w_3 are the corner displacements. The transformation

$$\bar{w}_{(x,y)} = w_{(x,y)} + Dx^3 + Mx^2y + Fxy^2 + Ny^3 \quad (14)$$

simplifies the third equation of equilibrium and Eqs. (10) also simplify to

$$\gamma_{xz} = (\partial w/\partial x) - EK_{1e}x^2/2G_{xz} + \nu_{zy}K_{1e}y^2 \quad (15a)$$

$$\gamma_{yz} = (\partial w/\partial y) - EK_{2e}y^2/2G_{yz} + \nu_{zx}K_{2e}x^2 \quad (15b)$$

Substituting Eqs. (13–15) into Eq. (12) and equating coefficients of δw_i on both sides of Eq. (12) yields the desired stiffness matrix. The resulting relation, as in the case of torsion,[9] is of the form

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{Bmatrix} = \frac{1}{4a}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ & & k_{33} \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} + \frac{E}{2a}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad (16)$$

where the k_{ij} coefficients are given on page 6 of Ref. 9, ($H = 0$) and C_i coefficients are

$$C_1 = -K_{1e}\left[\frac{3I_y}{2} - \frac{\nu_{zy}G_{xz}}{E}I_y - x_2M_y\right]Y_{23} + K_{1e}[I_{xy} - y_2M_y]X_{23} + K_{2e}\left[\frac{3I_x}{2} - \frac{\nu_{zx}G_{yz}}{E} \times I_y - y_2M_x\right]X_{23} - K_{2e}[I_{xy} - x_2M_x]Y_{23}$$

** For description and definitions of terms see Ref. 9.

$$C_2 = -K_{1e}\left(3I_y/2 - \frac{\nu_{zy}G_{xz}}{E}I_x - x_3M_y\right)Y_{31} + K_{1e}(I_{xy} - y_3M_y)X_{31} + K_{2e}\left(3I_x/2 - \frac{\nu_{zx}G_{yz}}{E} \times I_y - y_3M_x\right)X_{31} - K_{2e}(I_{xy} - x_3M_x)Y_{31}$$

$$C_3 = -K_{1e}\left(3I_y/2 - \frac{\nu_{zy}G_{xz}}{E}I_y - x_1M_y\right)Y_{12} + K_{1e}(I_{xy} - y_1M_y)X_{12} + K_{2e}\left(3I_x/2 - \frac{\nu_{zx}G_{yz}}{E} \times I_y - y_1M_x\right)X_{12} - K_{2e}(I_{xy} - x_1M_x)Y_{12}$$

The quantities M_x, M_y, I_x, I_y, I_{xy} refer to the area properties of the element. Having the stiffness matrix for the triangular element, a stiffness matrix for the arbitrary quadrilateral element of Fig. 4, Ref. 9, may be derived as indicated in that reference. Note that the bending and torsion matrices differ only in the vector C_i.

The procedure for solution of problems using this matrix, i.e., the derivation of any "problem matrix" and its solution, has been given in Ref. 9. Inhomogeneity is achieved by using different moduli of elasticity in appropriate elements.

4. Numerical Examples

The problem of a sandwich beam of four layers was analyzed. The inner layers had a modulus which was three times those of the two outer layers although the Poisson's ratio was the same in both. The shear stresses, parallel and perpendicular to the load, obtained for a 256-element break up were within 3% of those obtained from an exact stress function solution.

The next problem considered was that of a load applied along a diameter of a circular section of constant Poisson's ratio, and Young's modulus which varied in proportion to the absolute value of the distance perpendicular to the plane of loading. In the finite element method each element was assigned the average Young's modulus for that element. The finite element stresses compared favorably with the exact solution.[7]

Several other geometric shapes were analyzed and the results checked (within an acceptable error) wherever possible with exact solutions and acceptable approximate (thin-section) techniques.

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Turbulent Kinetic Energy Equation for a Transpired Turbulent Boundary Layer

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A few years ago Townsend[1] presented a discussion of several equilibrium turbulent boundary layers from the standpoint of the balance of the production and dissipation of turbulent kinetic energy in the various regions of the boundary layer. Among the cases considered was the particular case of a two-dimensional imcompressible turbulent boundary layer with uniform surface mass injection, which forms the subject of the present Note. Since that time, a considerable amount of work has been done on transpired turbulent boundary layers (Stevenson,[2] Kendall,[3] AlSaji,[4] Simpson, Moffat, and Kays,[5] Dahm and Kendall,[6] and Stevenson[7]) but the approach offered by Townsend has apparently not been exploited to date in studies of the inner region of an incompressible transpired turbulent boundary layer.

This Note presents experimental results correlated on the basis of the turbulent kinetic energy equation. These results indicate that the use of the turbulent kinetic energy equation provides additional information concerning the behavior of the turbulent boundary layer with surface mass injection.

The case considered is that of a uniform two-dimensional incompressible turbulent boundary layer developed initially over an impermeable surface with mass injected uniformly over the surface into the boundary layer at a downstream station. The axial pressure gradient is considered zero. For this case, the turbulent kinetic energy equation may be written as (Townsend[1] and Bradshaw et al.[8])

$$\langle\rho\rangle\langle U\rangle\partial(\langle q^2\rangle/2)/\partial x + \langle\rho\rangle\langle V\rangle\partial(\langle q^2\rangle/2)/\partial y + \langle\rho uv\rangle\partial\langle U\rangle/\partial y + \partial(\langle pv\rangle + \langle\rho q^2 v\rangle/2)/\partial y + \langle\rho\rangle\epsilon = 0 \quad (1)$$

where $\langle U\rangle$ and $\langle V\rangle$ are the mean velocities in the x and y directions, respectively, $\langle q^2\rangle/2 = (\langle u^2\rangle + \langle v^2\rangle + \langle w^2\rangle)/2$ is the turbulent kinetic energy, p is the pressure fluctuation, $\langle\rho\rangle$ is the mean density, u, v, and w are the velocity fluctuations, and ϵ is the dissipation of the turbulent kinetic energy due to viscous effects.

Equation (1) represents the rate of change of the turbulent kinetic energy as the net sum of the convection, production, diffusion, and viscous dissipation of the turbulent kinetic energy.

The incompressible turbulent kinetic energy equation may be converted into a shear stress equation by defining[1,8]

$$\tau/\langle\rho\rangle = -\langle\rho uv\rangle/\langle\rho\rangle \quad (2)$$

$$a_1 = \tau/(\langle\rho\rangle\langle q^2\rangle) \quad (3)$$

$$L = (\tau/\langle\rho\rangle)^{3/2}/\epsilon \quad (4)$$

and

$$(\langle pv\rangle/\langle\rho\rangle + \langle\rho q^2 v\rangle/2\langle\rho\rangle) = -a_2(\langle q^2\rangle^{3/2})\text{sign}(\partial\langle q^2\rangle/\partial y) \quad (5)$$

In these expressions a_1 and a_2 are taken as constants and L is an energy dissipation length. We should note that Townsend[1] introduced the expression used to model the diffusion term [Eq. (5)] from arguments concerning the structural equilibrium of the turbulence together with the sign of the gradient of the turbulent kinetic energy term. Bradshaw

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